Relaxation in the XX quantum chain

Thierry Platini and Dragi Karevski

Laboratoire de Physique des Matériaux, UMR CNRS 7556, Université Henri Poincaré, Nancy 1, BP 239, F-54506 Vandœuvre les Nancy Cedex, France

E-mail: karevski@lpm.u-nancy.fr

Abstract
We present the results obtained on the magnetization relaxation properties of an XX quantum chain in a transverse magnetic field. We first consider an initial thermal kink-like state where half of the chain is initially thermalized at a very high temperature T_b while the remaining half, called the system, is put at a lower temperature T_s. From this initial state, we derive analytically the Green function associated with the dynamical behaviour of the transverse magnetization. Depending on the strength of the magnetic field and on the temperature of the system, different regimes are obtained for the magnetic relaxation. In particular, with an initial droplet-like state, that is a cold subsystem of the finite size in contact at both ends with an infinite temperature environment, we derive analytically the behaviour of the time-dependent system magnetization.

PACS numbers: 75.40.Gb, 05.70.Ln

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Non-equilibrium properties of quantum chains generated by initial inhomogeneities, or by the sudden variation of an external field have been the purpose of several extended studies starting from the 1960s and continuing up to nowadays. In particular, in the context of free-fermionic models, that is Ising-like chains [1, 2], since the dynamics can be solved exactly [3, 4], a lot of exact results have been obtained. One may mention magnetization relaxation profiles which show a scaling behaviour [2, 5–13], two-point correlation functions characterizing ageing properties [10, 11], magnetization and energy current densities [14–18], and one may note that a quantized relaxation behaviour was also observed [19–21]. Since the very first studies of Niemejer and Tjion and others [22–25], the relaxation obtained shows typically a very slow decay. Indeed, they have shown that the relaxation of the magnetization in a spin-free fermionic chain after a sudden change of the external magnetic field is algebraic instead of

exponential [26] and that there is a lack of ergodicity in the sense that the magnetization does not relax towards its equilibrium value [24]. Consequently, in such systems there is no finite relaxation time. Moreover, it was shown that for certain inhomogeneously magnetized initial states, at special wave vectors, a slowing down of the magnetization relaxation can occur for gapped-free fermionic models [9]. The effect of quenched disorder was also considered in the Ising quantum chain [27]. Once more, for an initially completely ordered state, either in the field direction or perpendicular to it, an algebraic decay of the magnetization towards a stationary state was obtained. One may also mention a different approach in order to explore the relaxation to the equilibrium of quantum spins. It corresponds to the quantum version of the classical Kac ring model [28]. For example in [29], it was shown that in the thermodynamical limit the magnetization relaxes exponentially fast towards its equilibrium value. More recently, in the critical Ising quantum chain we have studied the relaxation of an initially magnetized finite droplet at a given temperature in contact at both ends with infinite Ising chains. We have shown that the droplet magnetization decays in time to the equilibrium magnetization with a $1/t f(x/t)$ behaviour [20], where $f(u)$ is a scaling function depending on the temperature. For a vanishing droplet initial temperature, the scaling function is simply given by the characteristic function $\Pi(u)$.

     In the present study, we focus our attention to the same situation in the *XX* quantum chain in a transverse magnetic field. Whereas in the Ising case, the transverse magnetization is not conserved in time, in the *XX* quantum chain it is a conserved dynamical quantity and one may expect that this conservation law will show up in a way or an other in the relaxation behaviour. To be more specific, let us precise the initial set-up. The system is initially prepared in a factorized canonical state $\rho(0) = \rho_b(0)\rho_s(0)$, where the density matrices $\rho_{b,s}(0) \propto e^{-\beta_{b,s}\mathcal{H}_{b,s}}$ are canonical states of the parts of the (infinite) chain each thermalized at a given temperature $\beta_{b,s}^{-1}$. At time $t = 0$, the (local) interactions between the different parts are switched on and the dynamics is generated by the total Hamiltonian $\mathcal{H} = \mathcal{H}_b + \mathcal{H}_s + \mathcal{H}_I$. The paper is organized as follows. In the following section we give some details of the by now classical diagonalization of the *XX* model and its dynamics. Section 3 gives more details on the initial set-up and on the time-dependent expectation values of physical quantities, putting a special focus on the transverse magnetization. In particular, it is shown that the time-dependent transverse magnetization may be expressed as a discrete convolution product of a Green function with the initial magnetization profile. In section 4, from an initial kink-like state we derive analytically the magnetization Green function and show that, at vanishing system temperature, one has to distinguish between the low- and high-field regimes. Consequently, in section 5 we show that the relaxation of the magnetization of an initial finite droplet of the size L_s, towards its equilibrium vanishing value, changes after an initial linear decay from an algebraic decay, L_s/t, at large fields to a faster size-independent decay at lower fields. However, this behaviour does not survive for finite system temperatures, but nevertheless the magnetization relaxation at low enough temperatures is strongly affected. We finally summarize our results and draw out some conclusions.

2. Diagonalization and dynamics

2.1. Diagonalization

The free boundary conditions XY quantum chain in a transverse field is defined through the Hamiltonian

$$\mathcal{H} = -\frac{1}{2} \sum_{n=1}^{L-1} \left[J_x \sigma_n^x \sigma_{n+1}^x + J_y \sigma_n^y \sigma_{n+1}^y \right] - \frac{h}{2} \sum_{n=1}^{L} \sigma_n^z, \tag{1}$$

where the σs are the Pauli matrices at site n, h is the transverse field applied in the z-direction and where we parametrize the coupling constants using the anisotropy factor $\kappa \in [-1, 1]$ as $J_x = (1+\kappa)/2$ and $J_y = (1-\kappa)/2$. In the special case $\kappa = 1$, the y-direction coupling constants vanish and one recovers the Ising quantum chain case. At $\kappa = 0$, $J_x = J_y$ and the Hamiltonian describes the *XX* model for which the total transverse magnetization $M^z = \sum_n \sigma_n^z$ is a conserved quantity since we have $[\mathcal{H}, M^z] = 0$. Consequently, the exchange interaction energy and the Zeeman energy are separately conserved quantities.

In terms of the lattice Clifford's operators Γ_n^1, Γ_n^2:

$$\Gamma_n^1 = \prod_{j=1}^{n-1} (-\sigma_j^z)\sigma_n^x, \qquad \Gamma_n^2 = -\prod_{j=1}^{n-1} (-\sigma_j^z)\sigma_n^y, \tag{2}$$

with $\Gamma_n^{i\dagger} = \Gamma_n^i$ satisfying the anticommutation rules

$$\{\Gamma_n^i, \Gamma_m^j\} = 2\delta_{i,j}\delta_{n,m}, \tag{3}$$

the Hamiltonian (1) takes the quadratic form [1, 2, 30]

$$\mathcal{H} = -\tfrac{1}{4}\boldsymbol{\Gamma}^\dagger \mathbf{T}\boldsymbol{\Gamma}, \tag{4}$$

where $\boldsymbol{\Gamma}$ is the $2L$-component vector:

$$\boldsymbol{\Gamma} = \begin{pmatrix} \Gamma^1 \\ \Gamma^2 \end{pmatrix}, \qquad \text{with} \quad \boldsymbol{\Gamma^i} = \begin{pmatrix} \Gamma_1^i \\ \Gamma_2^i \\ \vdots \\ \Gamma_L^i \end{pmatrix}, \qquad \text{and} \quad i = 1, 2. \tag{5}$$

The $2L \times 2L$ Hermitian matrix $\mathbf{T}$ is given by

$$\mathbf{T} = \begin{pmatrix} 0 & \mathbf{C} \\ \mathbf{C}^\dagger & 0 \end{pmatrix} \tag{6}$$

with

$$\mathbf{C} = -i \begin{pmatrix} h & J_y & & & 0 \\ J_x & h & J_y & & \\ & \ddots & \ddots & \ddots & \\ & & J_x & h & J_y \\ 0 & & & J_x & h \end{pmatrix}. \tag{7}$$

Parametrizing the eigenvectors V_q, associated with the eigenvalue ϵ_q, of $\mathbf{T}$ as

$$V_q = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_q \\ -i\psi_q \end{pmatrix}, \tag{8}$$

we obtain from the eigenvalue equation $\mathbf{T}V_q = \epsilon_q V_q$ the following equations:

$$-i\mathbf{C}\psi_q = \epsilon_q \phi_q, \qquad \mathbf{C}^\dagger \phi_q = -i\epsilon_q \psi_q. \tag{9}$$

We can note here that if we change simultaneously $\epsilon_q \rightarrow -\epsilon_q$ and $\psi_q \rightarrow -\psi_q$ these equations are unchanged. So, to each eigenvalue $\epsilon_q \geqslant 0$ corresponds an eigenvalue $\epsilon_{q'} = -\epsilon_q$ with the associated eigenvector $V_{q'}(\phi_{q'}, \psi_{q'}) = V_q(\phi_q, -\psi_q)$.

Introducing the fermionic creation and annihilation operators $\eta_q^\dagger$, η_q,

$$\eta_q^\dagger = \frac{1}{2} \sum_n \left(\phi_q(n)\Gamma_n^1 - i\psi_q(n)\Gamma_n^2\right)$$

$$\eta_q = \frac{1}{2} \sum_n \left(\phi_q(n)\Gamma_n^1 + i\psi_q(n)\Gamma_n^2\right), \tag{10}$$

satisfying the canonical Fermi–Dirac anticommutation rules $\{\eta_q^\dagger, \eta_{q'}\} = \delta_{q,q'}$, the Hamiltonian (1) takes the free-fermionic structure

$$\mathcal{H} = \sum_{q=1}^{L} \epsilon_q (\eta_q^\dagger \eta_q - 1/2), \tag{11}$$

where ϵ_q are the positive eigenvalue of **T**, given by

$$\epsilon_q = \sqrt{\kappa^2 \sin^2(q) + (h + \cos(q))^2}. \tag{12}$$

In the *XX* quantum chain, with $\kappa = 0$, one has for the (positive) spectrum

$$\epsilon_q = |h + \cos q| = |h - \cos k|, \tag{13}$$

with $k = \pi - q \in [0, \pi]$ and where the gap vanishes for $h \leqslant 1$ at $k_F = \arccos(h)$. In an equivalent picture, the (positive) spectrum associated with $k \in [0, k_F]$ is usually interpreted as corresponding to the creation of holes in the Fermi sea, while the ground state corresponds to $\prod_{k=0}^{k_F} b_k^\dagger |0\rangle$, where $b^\dagger$ are particle fermionic creation operators with negative energies $\varepsilon_k = -\epsilon_k$.

2.2. Dynamics

The Heisenberg equations of motion for the lattice Clifford's operators are easily solved, see the appendix 1 for more details, and lead to [2]

$$\Gamma_n^j(t) \equiv e^{i\mathcal{H}t} \Gamma_n^j e^{-i\mathcal{H}t} = \sum_{k,\nu} \langle \Gamma_k^\nu | \Gamma_n^j(t) \rangle \Gamma_k^\nu, \tag{14}$$

where $\langle C|D \rangle = \frac{1}{2}\{C^\dagger, D\}$ is a pseudo-scalar product with $\{.,.\}$ the anticommutator. The time-dependent contractions are explicitly given by

$$\langle \Gamma_k^1 | \Gamma_n^1(t) \rangle = \sum_q \phi_q(k)\phi_q(n) \cos(\epsilon_q t)$$

$$\langle \Gamma_k^2 | \Gamma_n^2(t) \rangle = \sum_q \psi_q(k)\psi_q(n) \cos(\epsilon_q t) \tag{15}$$

$$\langle \Gamma_k^1 | \Gamma_n^2(t) \rangle = \langle \Gamma_n^2 | \Gamma_k^1 \rangle_{-t} = -\sum_q \phi_q(k)\psi_q(n) \sin(\epsilon_q t).$$

In the *XX* chain case, in the thermodynamic limit they take the form

$$\langle \Gamma_k^1 | \Gamma_n^1(t) \rangle = i^{n-k} J_{n-k}(t) \times \begin{cases} \cos(ht) & \forall \, (n-k) \text{ even} \\ -i\sin(ht) & \forall \, (n-k) \text{ odd} \end{cases} \tag{16}$$

$$\langle \Gamma_k^1 | \Gamma_n^2(t) \rangle = i^{n-k} J_{n-k}(t) \times \begin{cases} -\sin(ht) & \forall \, (n-k) \text{ even} \\ -i\cos(ht) & \forall \, (n-k) \text{ odd}, \end{cases} \tag{17}$$

with $\langle \Gamma_k^2 | \Gamma_n^2(t) \rangle = \langle \Gamma_k^1 | \Gamma_n^1(t) \rangle$ and where $J_n(t)$ is the *J*-Bessel function of order n. These contractions are the basic time-dependent quantities of the *XX* chain.

Formally, since $\langle \Gamma_k^i | \Gamma_n^j \rangle = \delta_{i,j} \delta_{n,k}$, the set of $\{\Gamma_k^i\}$ forms an orthonormal basis of a $2L$-dimensional linear vector space $\mathcal{E}$ with an inner product defined by $\langle .|. \rangle = \frac{1}{2}\{.^\dagger, .\}$. Hence, every vector $X \in \mathcal{E}$ has a unique expansion $X = \sum_{i,k} \langle \Gamma_k^i | X \rangle \Gamma_k^i$. A string operator of the form $\mathcal{O} \equiv X_1 X_2 \ldots X_n$ is a direct product vector in $\mathcal{E}^{\otimes n}$ and has the expansion

$$\mathcal{O} \equiv X_1 X_2 \ldots X_n = \sum_{i_1,k_1} \ldots \sum_{i_n,k_n} \langle \Gamma_{k_1}^{i_1} | X_1 \rangle \ldots \langle \Gamma_{k_1}^{i_n} | X_n \rangle \Gamma_{k_1}^{i_1} \ldots \Gamma_{k_n}^{i_n}. \tag{18}$$

3. Initial conditions and expectation values

3.1. Product initial state

The system is initially prepared in a factorized state (a product measure state) [10, 12, 13, 20, 21] of the form

$$\rho(0) = \prod_j \rho_j(0), \tag{19}$$

where the density matrix

$$\rho_j(0) = \frac{1}{Z_j} \exp(-\beta_j \mathcal{H}_j) \tag{20}$$

is a canonical state, at inverse temperature β_j, associated with the Hamiltonian $\mathcal{H}_j$ of the jth subsystem of the total chain decomposed in the following way:

$$\mathcal{H} = \sum_j \mathcal{H}_j + \sum_j \mathcal{H}^I_{j,j+1}, \tag{21}$$

where the Pauli matrices of the jth part are written $(\sigma_n)_j$. The interface interaction term is

$$\mathcal{H}^I_{j,j+1} = -\tfrac{1}{4}\big[(\sigma^x_{L_j})_j(\sigma^x_1)_{j+1} + (\sigma^y_{L_j})_j(\sigma^y_1)_{j+1}\big], \tag{22}$$

where we wrote the Pauli matrices of the jth part as $(\sigma_n)_j$.

3.2. Expectation value of the transverse magnetization

The expectation value of an observable $\mathcal{O}$ at a time t is given by

$$\langle \mathcal{O}(t) \rangle = \text{Tr}\left\{ \mathcal{O}(t) \prod_j \rho_j(0) \right\}, \tag{23}$$

where $\mathcal{O}(t) = e^{i\mathcal{H}t} \mathcal{O} e^{-i\mathcal{H}t}$ is the operator associated with the observable at time t in the Heisenberg picture. In particular if $\mathcal{O}$ is a string of Clifford's operators $\mathcal{O} = \Gamma^{j_1}_{l_1} \ldots \Gamma^{j_n}_{l_n}$, one has at time t

$$\langle \mathcal{O}(t) \rangle = \sum_{i_1, k_1} \ldots \sum_{i_n, k_n} \langle \Gamma^{i_1}_{k_1} | \Gamma^{j_1}_{l_1}(t) \rangle \ldots \langle \Gamma^{i_n}_{k_n} | \Gamma^{j_n}_{l_n}(t) \rangle \, \text{Tr}\{ \Gamma^{i_1}_{k_1} \ldots \Gamma^{i_n}_{k_n} \rho(0) \}, \tag{24}$$

where the elements $\text{Tr}\{ \Gamma^{i_1}_{k_1} \ldots \Gamma^{i_n}_{k_n} \rho(0) \}$ represent the relevant initial state information (correlations) for the time evolution of the quantity $\mathcal{O}$. Since the time contractions are explicitly known one needs only to compute the initial state properties of the string operators for a given $\rho(0)$.

In this work we focus our attention on the time behaviour of the transverse magnetization which is, as already stated, a conserved quantity. The transverse magnetization is one of the simplest Clifford string operators. Namely, it takes the form $\sigma^z_n = -i\Gamma^2_n \Gamma^1_n$ and at time t, in the Heisenberg picture we have

$$\sigma^z_n(t) = -i\Gamma^2_n(t)\Gamma^1_n(t). \tag{25}$$

Its expectation value, in the thermodynamic limit, at time t is given by

$$\langle \sigma^z_n \rangle(t) = \sum_{k,j} F^j_t(n,k) I_{k,k+j}, \tag{26}$$

where $I_{k,k+j} = \text{Tr}\{-i\Gamma_k^2 \Gamma_{k+j}^1 \rho(0)\}$ is the initial condition matrix. The time-dependent functions F_t^j are given by

$$F_t^j(n,k) = \langle \Gamma_k^1 | \Gamma_n^1(t)\rangle\langle \Gamma_{k+j}^2 | \Gamma_n^2(t)\rangle - \langle \Gamma_k^1 | \Gamma_n^2(t)\rangle\langle \Gamma_{k+j}^2 | \Gamma_n^1(t)\rangle, \qquad (27)$$

and they can be explicitly evaluated in the thermodynamic limit:

$$F_t^{2j}(n,k) = F_t^{2j}(n-k) = (-1)^j J_{n-k}(t) J_{n-k-2|j|}(t), \qquad (28)$$

and they identically vanish for odd j. Consequently, we have for the time behaviour of the transverse magnetization

$$\langle \sigma_n^z \rangle(t) = \sum_{j=-\infty}^{\infty} \left(F_t^{2j} \star P^{2j}\right)(n), \qquad (29)$$

where $\star$ stands for a discrete convolution product and where we introduced the notations $P^{2j}(k) = I_{k,k+2j}$.

One needs now to evaluate the initial matrix I. First remark that for the Clifford operators Γ belonging to different initial blocks, the trace identically vanishes, reflecting the fact that they are traceless operators and that there are no initial correlation between them since the initial state is of a product form. So, the only non-vanishing contribution comes whenever the two Γs are in the same subsystem. Consequently, the I matrix has a diagonal form such that

$$I = \begin{pmatrix} \ddots & 0 & & & \\ 0 & I_{l-1} & 0 & & \\ & 0 & I_l & 0 & \\ & & 0 & \ddots & \end{pmatrix}, \qquad (30)$$

where I_l is an $L_l \times L_l$ matrix with L_l being the size of the lth subsystem, with matrix elements

$$(I_l)_{k,k'} = -\sum_{q=1}^{L_l} \psi_q^l(k)\phi_q^l(k') \tanh\left(\frac{\beta_l \epsilon_q^l}{2}\right), \qquad (31)$$

where the l-labelled quantities refer to the lth subsystem Hamiltonian $\mathcal{H}_l$.

4. System in contact with an infinite temperature 'bath'

4.1. Transverse magnetization green function

We start our analysis by considering a chain where half of it (the left) is thermalized at inverse temperature β_b, and is called the bath, and where the right half, the system, is initially at inverse temperature $\beta_s \equiv \beta$. Consequently, the initial matrix $\mathbf{I}$ is

$$\mathbf{I} = \begin{pmatrix} \mathbf{I}_b & 0 \\ 0 & \mathbf{I}_s \end{pmatrix}. \qquad (32)$$

Taking the infinite bath temperature limit, the initial matrix $\mathbf{I}_b$ vanishes so that only the system part remains with

$$(I_s)_{k,k'} = -\sum_{q=1}^{L_s} \psi_q^s(k)\phi_q^s(k') \tanh\left(\frac{\beta \epsilon_q^s}{2}\right), \qquad (33)$$

where L_s is the system size. The system initial matrix can be decomposed into a surface contribution, near the interface boundary, and a homogeneous bulk part

$$(\mathbf{I_s})_{k,k'} = \left(\mathbf{I_s}^{\text{Bulk}}\right)_{k,k'} + \left(\mathbf{I_s}^{\text{Surf}}\right)_{k,k'}, \qquad (34)$$

with

$$\left(\mathbf{I_s}^{\text{Bulk}}\right)_{k,k'} = \sum_{n=1}^{L_s} \frac{(-1)^{k+k'}}{L+1} \cos(q_n(k-k')) \tanh(\beta\epsilon_{q_n}/2) \tag{35}$$

$$\left(\mathbf{I_s}^{\text{Surf}}\right)_{k,k'} = -\sum_{n=1}^{L_s} \frac{(-1)^{k+k'}}{L+1} \cos(q_n(k+k')) \tanh(\beta\epsilon_{q_n}/2). \tag{36}$$

In the large size limit, $L_s \to \infty$,

$$\left(\mathbf{I_s}^{\text{Bulk}}\right)_{k,k'} = (-1)^{k+k'} \int_0^\pi \frac{d\theta}{\pi} \cos((k-k')\theta) \tanh(\beta\epsilon_h(\theta)), \tag{37}$$

and

$$\left(\mathbf{I_s}^{\text{Surf}}\right)_{k,k'} = -(-1)^{k+k'} \int_0^\pi \frac{d\theta}{\pi} \cos((k+k')\theta) \tanh(\beta\epsilon_h(\theta)/2), \tag{38}$$

for the surface contribution, with $\epsilon_h(\theta) = h - \cos(\theta)$.

4.2. Zero temperature limit

In the system low temperature limit, $T_s = 0$, one has to distinguish between the regions $h < 1$ and $h \geqslant 1$ since $\tanh(\beta\epsilon_h(\theta)/2) = \text{sgn}(\epsilon_h(\theta))$.

(i) $h \geqslant 1$ *region.* For $h \geqslant 1$, the excitations are always positive and the sum in (33) reduces to $-\sum_q \psi_q^s(k)\phi_q^s(k') = \delta_{k,k'}$, so that the initial matrix is just the identity matrix. The average magnetization, in the thermodynamic limit $L_{s,b} \to \infty$ is then given at later time by

$$m^z(n,t) \equiv \langle\sigma_n^z\rangle(t) = \sum_{k=-\infty}^{\infty} F_t^0(n-k)H(k), \tag{39}$$

where H is the Heaviside function. The Green function $G_t(n) = F_t^0(n) = J_n^2(t)$ can be expressed in the continuum limit as [12] $G_t(n) \sim \frac{1}{t}g(n/t)$ with the scaling function

$$g(u) = \frac{1}{\pi\sqrt{1-u^2}}, \qquad |u| \leqslant 1 \tag{40}$$

and zero otherwise. Finally, with the step-like initial condition we have in the continuum limit

$$m^z(x,t) = \begin{cases} 0 & x < -t \\ \dfrac{1}{\pi} \arcsin(x/t) + \dfrac{1}{2} & -t \leqslant x \leqslant t \\ 1 & t < x. \end{cases}$$

This solves the extreme temperature set-up for $h \geqslant 1$.

(ii) $h < 1$ *region.* For $h < 1$, one has a k-independent profile for the bulk contribution:

$$p^j \equiv \left(\mathbf{I_s}^{\text{Bulk}}\right)_{k,k+j} = (-1)^{j+1}\frac{2}{\pi}\frac{\sin(j\theta_F)}{j} \qquad \forall\, j \neq 0, \tag{41}$$

where $\theta_F = \arccos(h)$, and for the diagonal part, $j = 0$,

$$p^0 \equiv m^z(0) \equiv \left(\mathbf{I_s}^{\text{Bulk}}\right)_{k,k} = \frac{2}{\pi} \arcsin(h). \tag{42}$$

The surface term leads to a k-decaying part

$$\left(\mathbf{I_s}^{\text{Surf}}\right)_{k,k+j} = (-1)^j\frac{2}{\pi}\frac{\sin((2k+j)\theta_F)}{2k+j} \qquad \forall\, j. \tag{43}$$
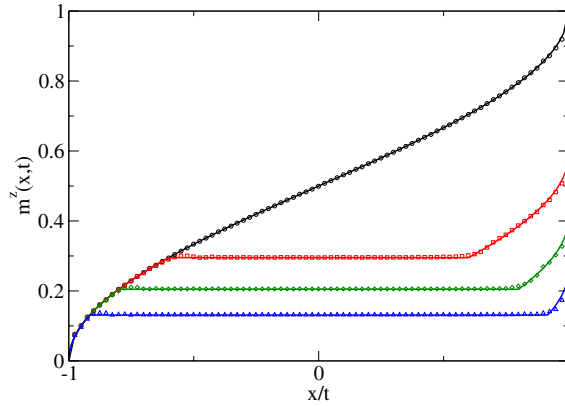
Figure 1. Magnetization profiles with $T_s = 0$, $T_b = \infty$ for different fields $h = 1$, $h = 0.8$, $h = 0.6$ and $h = 0.4$ from top to bottom on a chain of total size $L = 500$. The analytical results are in full lines.

Neglecting the boundary effect term, I^{Surf}, and using the fact that $F_t^{2j+1}(n) = 0$, one has for the transverse magnetization

$$\langle \sigma_n^z \rangle(t) = \left(\left[\sum_j \frac{p^{2j}}{m^z(0)} F_t^{2j} \right] \star m^z(0) H \right)(n) = (G_t \star m^z(0))(n), \qquad (44)$$

where we have used the fact that $(\mathbf{I}^{\text{Bulk}})_{k,k+2j} = P^{2j}(k) = p^{2j} H(k)$. The Green function G_t is determined by the spatial derivative of the former equation, since $[m^z(x,t)]' = G_t \star m^z(0) H' = m^z(0) G_t \star \delta = m^z(0) G_t$. In the asymptotic limit (see the appendix) one obtains

$$G_t(x) = \frac{1}{t} g\left(\frac{x}{t}\right) = \frac{1}{t} \frac{1}{2 \arcsin(h)\sqrt{1 - (x/t)^2}}, \qquad \sqrt{1 - h^2} \leqslant |\frac{x}{t}| \leqslant 1, \qquad (45)$$

and zero otherwise[1]. This disconnected support of the Green function will play an important role when considering the relaxation of a finite initially magnetized domain. It will lead to the appearance of a finite relaxation time.

It turns out that the step-like initial state generated by the two-temperature set-up develops a plateau of constant magnetization, as it was already observed in [16, 13], due to the vanishing of the Green function in the domain $|x/t| < \sqrt{1 - h^2}$:

$$m^z(x, t) = \Phi_h(x/t) + m^z(0)/2 \qquad (46)$$

with $\Phi_h(-u) = -\Phi_h(u)$, where

$$\Phi_h(u) = \begin{cases} 0 & 0 < u \leqslant \sqrt{1 - h^2} \\ m^z(0)/2 - \arccos(u)/\pi & \sqrt{1 - h^2} < u \leqslant 1 \\ m^z(0)/2 & 1 < u \end{cases}.$$

We show the numerical and analytical results in figure 1 for different fields.

---

[1] Note that the same analysis is handy in the Ising case at critical transverse field. The result is simpler with a Green function given by $G_t(x) = \frac{1}{t} g^{\text{Ising}}(\frac{x}{t}) = \frac{1}{2t} \Pi(\frac{x}{2t})$ where $\Pi(u) = 1 \ \forall \ u \in [-1/2, 1/2]$ and zero otherwise. Details of the derivation are given in appendix C.
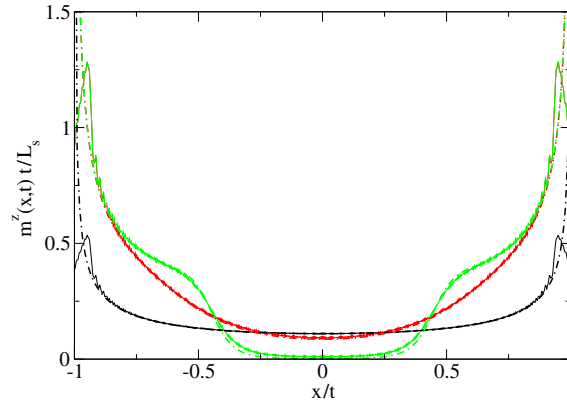
Figure 2. Numerical rescaled magnetization profiles for three different inverse temperatures (At $x/t = 0$ from top to bottom $\beta_s = 1$, $\beta_s = 10$, $\beta_s = 100$) at $h = 0.8$. The size of the system is $L_s = 100$ and the time is $t = 700$. The dashed lines are the corresponding analytical curves for the Green functions.

4.3. Finite temperature effects

While keeping a very hot bath (formally taken to be at infinite temperature) we now consider the effect of a finite initial temperature system state on the relaxation of the transverse magnetization. As in the previous section, we can compute the Green function in the asymptotic limit from the step-like initial condition. One finds for $h \geqslant 1$ (see the appendix for details) after a lengthy derivation

$$G_t(x) = \frac{1}{2\pi t m^z(0)\sqrt{1 - \left(\frac{x}{t}\right)^2}} \left\{ \tanh\frac{\beta_s}{2}\left(h + \sqrt{1 - \left(\frac{x}{t}\right)^2}\right) + \tanh\frac{\beta_s}{2}\left(h - \sqrt{1 - \left(\frac{x}{t}\right)^2}\right)\right\},$$

(47)

where β_s is the inverse initial system temperature and where

$$m^z(0) = \frac{1}{\pi}\int_0^\pi d\theta \tanh\left(\frac{\beta_s}{2}(h - \cos\theta)\right)$$

(48)

is the initial bulk magnetization density. Expression (47) is also valid at $h < 1$. In figure 2, we have plotted the numerical magnetization profiles compared with the analytical result (47) for different temperatures, since in the long time limit $t \gg L_s$ one recovers the Green function as $m^z(x, t) = G_t(x) \star \frac{m^z(0)L_s}{t}\delta(x/t)$.

One may note that it is also possible to deduce the Green function in a very simple way starting with the kink-like initial state and assuming that the magnetization at the interface takes for all times its asymptotic value $m^z(0)/2$, since the initial bath magnetization is zero while the initial system magnetization is $m^z(0)$. Indeed, using (44), which is in fact also valid at $h \geqslant 1$, one has $m^z(x, t) = (G_t \star m^z(0)H)(x)$. Together with the scaling assumption $G_t(x) = \frac{1}{t m^z(0)}g(x/t)$ one has

$$m^z(x, t) = \int_{-1}^x du\, g(u).$$

(49)

Now, taking into account the fact that at the interface $x = 0$, the magnetization is given by (48), one has

$$m^z(0, t) = \int_{-1}^0 du\, g(u) = \frac{m^z(0)}{2} = \int_0^\pi \frac{d\theta}{2\pi}\tanh\left(\frac{\beta_s}{2}(h - \cos\theta)\right).$$

(50)

Decomposing the integral into two pieces, one from 0 to $\pi/2$ and the other one from $\pi/2$ to π, by a simple change of the integration variable one obtains

$$\int_{-1}^{0} \mathrm{d}u\, g(u) = \int_{-1}^{0} \mathrm{d}u\, \frac{\tanh\left(\frac{\beta_s}{2}(h - \sqrt{1 - u^2})\right) + \tanh\left(\frac{\beta_s}{2}(h + \sqrt{1 - u^2})\right)}{2\pi\sqrt{1 - u^2}}, \tag{51}$$

which gives the identification $g(u)$ and so $G_t(x)$.

5. Relaxation of an initial droplet

If one considers initially a finite system of the size L_s (the initial droplet) at an inverse temperature β_s in contact at both ends with infinite temperature 'baths', from the convolution of the asymptotic Green function with $P^0(k) = m^z(0)\Pi\left(\frac{k}{L_s}\right)$ one obtains the transverse magnetization at later times. The total droplet magnetization density $m^z(t) = \frac{1}{L_s}\sum_{k \in L_s} m^z(k, t)$ will relax towards zero since the baths are at infinite temperature (and consequently have a vanishing transverse magnetization). From the asymptotic Green function obtained in the different regimes $h \geqslant 1$ and $h < 1$ we obtain two different relaxation behaviour.

5.1. Zero temperature droplet

In the $h \geqslant 1$ regime, the droplet magnetization density behaves as

$$m^z(t) = \begin{cases} 1 - \dfrac{2}{\pi}\dfrac{t}{L_s} & t \leqslant \tau_1 \\[4mm] \dfrac{2}{\pi}\arcsin\left(\dfrac{L_s}{t}\right) - \dfrac{2}{\pi}\dfrac{t}{L_s}\left(1 - \sqrt{1 - \left(\dfrac{L_s}{t}\right)^2}\right) & t \geqslant \tau_1, \end{cases} \tag{52}$$

with $\tau_1 = L_s$. Asymptotically the magnetization relaxes with a power law behaviour [10]:

$$m^z(t) \simeq \frac{L_s}{\pi t} \qquad h \geqslant 1. \tag{53}$$

One may note that in the critical Ising chain one has exactly the same relaxation behaviour for the droplet magnetization: $m^z(t) \simeq \frac{L_s}{\pi t}$ [20, 21].

In contrast, in the low-field regime, $h < 1$, the time behaviour of the droplet magnetization density obtained from the asymptotic Green function (45) is given by

$$m^z(t) = \frac{2}{\pi}\arccos(\sqrt{1 - h^2}) - \frac{2}{\pi}\frac{t}{L_s}h, \qquad t \leqslant \tau_1$$

$$m^z(t) = \frac{2}{\pi}\arccos(\sqrt{1 - h^2}) - \frac{2}{\pi}\arccos\left(\frac{L_s}{t}\right) \tag{54}$$

$$-\frac{2}{\pi}\frac{t}{L_s}\left(h - \sqrt{1 - \left(\frac{L_s}{t}\right)^2}\right), \qquad \tau_1 \leqslant t \leqslant \tau,$$

and

$$m^z(t) = 0, \qquad t \geqslant \tau_2, \tag{55}$$

with $\tau_2 = L_s/\sqrt{1 - h^2} = L_s/\cos(m^z(0)\pi/2)$. The vanishing of the magnetization at times $t \geqslant \tau_2$ is a direct consequence of the vanishing of the Green function in the region $-\sqrt{1 - h^2} < x/t < \sqrt{1 - h^2}$. However, this exact cancellation is valid only in the limit $L_s \to \infty$. For a finite droplet, one has finite size corrections to the Green function such that at the leading order the Green function is given by $G_t(x) = g(x/t)/t[1 + \mathcal{O}(1/L_s)]$. These
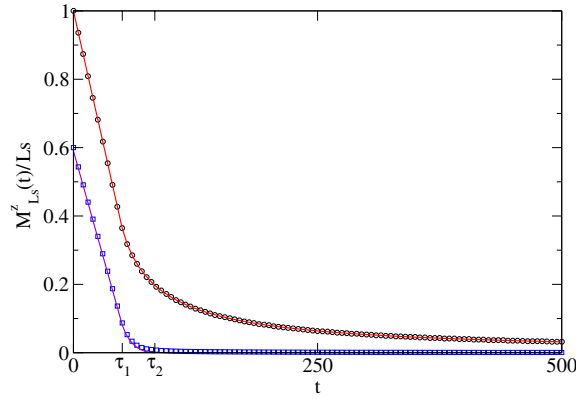
Figure 3. Relaxation of the magnetization density at fields, from top to bottom, $h = 1$ and $h = 0.8$ for an initial droplet of size $L_s = 50$. The full lines correspond to the analytical results.

corrections lead for the magnetization density to a size-independent correction to zero of order $1/t$. Consequently, the ratio between the magnetization densities at $h < 1$ and $h \geqslant 1$ is asymptotically (in time) given by

$$\frac{m^z_{h<1}(t)}{m^z_{h\geqslant 1}(t)} = \mathcal{O}\left(\frac{1}{L_s}\right). \tag{56}$$

It is in this sense that the magnetization density within the droplet changes from a slow relaxation (53), depending on the size of the system, to a much faster relaxation since it is independent of the size. If one considers the total remaining magnetization of the droplet, we pass from a diffusive like behaviour L_s^2/t at $h \geqslant 1$ to a ballistic-like behaviour L_s/t at $h < 1$. These two regimes reflect the fact that the dispersion relation changes from a parabolic shape, $\epsilon(k) \sim h - 1 + k^2/2$ at $h \geqslant 1$ to a linear regime $\epsilon(k) \sim |k - k_F|$ for $h < 1$.

In figure 3, we show the numerical results compared to the analytical expressions obtained for the droplet magnetization density. The agreement between the exact numerical results and the analytical asymptotic expressions is excellent.

5.2. Finite temperature droplet

With the help of the finite temperature Green function (47), one may compute the droplet magnetization density at the leading order:

$$m^z(t) \simeq \frac{L_s}{\pi t} \frac{1}{2} \left(\tanh\left[\frac{\beta_s}{2}(h + 1)\right] + \tanh\left[\frac{\beta_s}{2}(h - 1)\right] \right) \tag{57}$$

which shows the L_s/t power law decay for any transverse field h. In the $h < 1$ region, the faster relaxation is recovered in the low temperature limit $\beta_s \to \infty$. Indeed, for sufficiently small temperature the magnetization density is given by

$$m^z(t) \simeq \frac{L_s}{\pi t} \exp(-\beta_s(1 - h)), \tag{58}$$

and one may identify a typical size-dependent inverse temperature $\beta(L_s) \equiv \ln L_s/(1 - h)$ and one recovers the fast relaxation for inverse temperatures $\beta_s \gg \beta(L_s)$.

6. Summary and discussion

We have presented here analytical results, confirmed by exact numerical calculations, of the relaxation of the transverse magnetization of the XX quantum chain. In particular, starting from an initial product state depending on the strength of the initial magnetic field, we have obtained the asymptotic Green function associated with the transverse magnetization. When the system is divided initially into two parts, a hot bath and a cold subsystem, we have shown that the Green function takes the scaling form $G_t(x) = \frac{1}{t} g(x/t)$, where the explicit shape of the scaling function $g(u)$ depends on the initial temperatures and on the field. In the extreme scenario, with a vanishing subsystem temperature and an infinite bath temperature, at low fields, $h < 1$, the scaling function shows the very peculiarity that its support is disconnected. This property leads to the emergence of a flat magnetization profile when one starts with a kink-like initial profile [16]. In the case of an initial finite magnetized domain it leads to the fact that its relaxation is much faster in the low-field regime, $h < 1$, than it is at higher fields. Namely, one has $m_{h<1}(t)/m_{h \geqslant 1}(t) \sim 1/L_s$, where $m(t)$ is the domain magnetization density at time t and where L_s is the size of the domain.

Finally, one may note that for a Green function taking the scaling form $G_t(x) = \frac{1}{t} g(x/t)$, one has identically

$$\partial_t G + \partial_x \frac{x}{t} G = 0, \tag{59}$$

so that the impulsional response magnetization $m_i^z = m^z(0) G_t$ decays in time and satisfies the continuity equation

$$\partial_t m_i^z(x, t) + \partial_x j_i(x, t) = 0, \tag{60}$$

with the current j_i simply given by $j_i(x, t) = \frac{x}{t} m_i^z(x, t)$. In the step-like situation, the scaling form $\frac{1}{t} g(x/t)$ of the Green function leads to a scaling solution $m_s^z(x, t) = \Phi(x/t)$ which obviously satisfies the homogeneous scaling equation

$$(t \partial_t + x \partial_x) m_s^z(x, t) = 0, \tag{61}$$

which leads for the current j_s to the equation $\partial_x j_s = \frac{x}{t} \partial_x m_s^z$. So that the step-like current is related to the impulsional current through $\partial_x j^s = j^i$ since $\partial_x m_s^z = m_i^z$.

Acknowledgments

Appendix A. Dynamics of the lattice operators

The Clifford operators Γ_n^1, Γ_n^2 can be expressed in terms of fermionic operators:

$$\Gamma_n^1(t) = \sum_{q=1}^{L} \phi_q(n) \big(\eta_q^\dagger(t) + \eta_q(t) \big)$$
$$\Gamma_n^2(t) = \sum_{q=1}^{L} i \psi_q(n) \big(\eta_q^\dagger(t) - \eta_q(t) \big). \tag{A.1}$$

The time evolution of the creation and annihilation operators is obtained by $\eta_q^\dagger(t) = U_q^\dagger(t)\eta_q^\dagger U_q(t)$ with

$$U_q(t) = \mathrm{e}^{-\mathrm{i}\epsilon_q \eta_q^\dagger \eta_q t} \tag{A.2}$$

and leads to

$$\begin{aligned}
\eta_q^\dagger(t) &= \mathrm{e}^{\mathrm{i}\epsilon_q t}\eta_q^\dagger \\
\eta_q(t) &= \mathrm{e}^{-\mathrm{i}\epsilon_q t}\eta_q.
\end{aligned} \tag{A.3}$$

With the help of these relations we can re-express the Clifford operators in terms of the initial time operator Γ. Finally, one obtains

$$\Gamma_n^j(t) = \sum_{k,i} \langle \Gamma_k^i | \Gamma_n^j \rangle_t \Gamma_k^i, \tag{A.4}$$

with

$$\begin{aligned}
\langle \Gamma_k^1 | \Gamma_n^1 \rangle_t &= \sum_q \phi_q(k)\phi_q(n)\cos(\epsilon_q t) \\
\langle \Gamma_k^2 | \Gamma_n^2 \rangle_t &= \sum_q \psi_q(k)\psi_q(n)\cos(\epsilon_q t) \\
\langle \Gamma_k^1 | \Gamma_n^2 \rangle_t &= \langle \Gamma_n^2 | \Gamma_k^1 \rangle_{-t} = -\sum_q \phi_q(k)\psi_q(n)\sin(\epsilon_q t).
\end{aligned} \tag{A.5}$$

Appendix B. Zero temperature *XX* Green function

The contribution of the bulk to the magnetization is given by

$$m^z(n,t) = \sum_j \sum_k F_t^j(n-k)\left(\mathbf{I_s}^{\mathrm{Bulk}}\right)_{k,k+j}. \tag{B.1}$$

Using $\left(\mathbf{I_s}^{\mathrm{Bulk}}\right)_{k,k+j} = H(k)p^j$ and the relation $\left(\mathbf{I_s}^{\mathrm{Bulk}}\right)_{k,k+j} = \left(\mathbf{I_s}^{\mathrm{Bulk}}\right)_{k+j,k}$ one has

$$m^z(n,t) = m^z(0)\sum_{k=0}^\infty J_{n-k}^2(t) + 2\sum_{k=0,j=1}^\infty (-1)^j J_{n-k}(t)J_{n-k-2j}(t)p^{2j}, \tag{B.2}$$

with $p^0 = m^z(0)$. The spatial derivative $\Phi'(n+1,t) = m^z(n+1,t) - m^z(n,t)$ is

$$\Phi'(n,t) = J_n(t)\left(m^z(0)J_n(t) + 2\sum_{j=1}^\infty (-1)^j p^{2j} J_{n-2j}(t)\right). \tag{B.3}$$

Using the asymptotic expansion of the Bessel function for $\nu \gg 1$:

$$J_n(t) = J_\nu\left(\frac{\nu}{\cos\beta}\right) = \sqrt{\frac{2}{\pi\nu\tan\beta}}\cos\psi, \tag{B.4}$$

with $\psi = \nu(\tan\beta - \beta) - \pi/4$, and taking $n \to \infty$ one has

$$\begin{aligned}
\Phi'(n,t) \simeq \frac{2}{\pi t\sqrt{1 - \left(\frac{n}{t}\right)^2}} &\left\{\cos^2(\psi)\left[m^z(0) + 2\sum_{j\geqslant 1}(-1)^j p^{2j}\cos\left(2j\arccos\left(\frac{n}{t}\right)\right)\right]\right. \\
&\left. - \cos(\psi)\sin(\psi)\left[2\sum_{j\geqslant 1}(-1)^j p^{2j}\sin\left(2j\arccos\left(\frac{n}{t}\right)\right)\right]\right\}.
\end{aligned} \tag{B.5}$$

Keeping only the first term in this expression, where $\cos^2(\psi)$ is replaced by its average value $1/2$, and taking the explicit expressions for the profiles p^j, given in (41), together with

$$\sum_{j=1}^{\infty}(-1)^j \frac{\sin(jx)}{j} = -x/2, \qquad \forall x \in]-\pi, \pi[, \tag{B.6}$$

$$\sum_{j=1}^{\infty}(-1)^j \frac{\sin(jx)}{j} = \pi - x/2, \qquad \forall x \in]\pi, 3\pi[, \tag{B.7}$$

one obtains finally

$$\Phi'(n,t) = \frac{1}{\pi t \sqrt{1-v^2}} \begin{cases} 0 & n/t \leqslant \sqrt{1-h^2} \\ 1 & \sqrt{1-h^2} < n/t \leqslant 1, \end{cases}$$

for positive n and $\Phi'(-n,t) = \Phi'(n,t)$. The Green function is simply given by $G(x,t) = \Phi'(x,t)/m^z(0)$.

Appendix C. Critical Ising Green function

In the Ising case the bulk contribution is given by (see [20, 21])

$$m^z(n,t) = \sum_j \sum_k P_n^{k,k+j}(t)\left(\mathbf{I}_s^{\text{Bulk}}\right)_{k,k+j}, \tag{C.1}$$

with

$$(-1)^j P_n^{k,k+j}(t) = J_{2(n-k)}(2t)J_{2(n-k)-2j}(2t) - J_{2(n-k)+1}(2t)J_{2(n-k)-2j-1}(2t). \tag{C.2}$$

At system vanishing temperature $T_s = 0$ the I_s elements are given by

$$\left(\mathbf{I_s}^{\text{Bulk}}\right)_{k,k'} = \frac{2}{\pi}\frac{(-1)^{k'-k}}{2(k'-k)+1}. \tag{C.3}$$

Using the relation $\left(\mathbf{I_s}^{\text{Bulk}}\right)_{k,k'} = \left(\mathbf{I_s}^{\text{Bulk}}\right)_{k'+1,k}$ one has

$$m^z(n,t) = \sum_{j=0}^{\infty}\sum_k F_t^j(n-k)(\mathbf{I}^B)_{k,k+j}, \tag{C.4}$$

with

$$(-1)^j F_t^j(p) = 2(J'_{2p}(2t)J_{2p-2j-1}(2t) - J_{2p}(2t)J'_{2p-2j-1}(2t)). \tag{C.5}$$

The spatial derivative function $\Phi'(n+1,t) = m^z(n+1,t) - m^z(n,t)$ is then given by

$$\Phi'(n,t) = \frac{4}{\pi}J'_{2n}(2t)\sum_{j=0}^{\infty}\frac{J_{2n-2j-1}(2t)}{2j+1} - \frac{4}{\pi}J_{2n}(2t)\sum_{j=0}^{\infty}\frac{J'_{2n-2j-1}(2t)}{2j+1}. \tag{C.6}$$

Taking this expression together with the asymptotic expansion of the Bessel function we finally arrive at

$$G(x,t) = \frac{1}{2t}\Pi\left(\frac{x}{2t}\right). \tag{C.7}$$

Appendix D. Finite temperature *XX* Green function

In the system non-zero temperature case, one has

$$\Phi'(n, t) = m^z(0) J_n^2(t) + J_n(t) A_n(t), \tag{D.1}$$

with

$$A_n(t) = 2 \sum_{j=1}^{\infty} (-1)^j p^{2j} J_{n-2j}(t), \tag{D.2}$$

where $p^{2j} H(k) = (I^{\text{Bulk}})_{k,k+2j}$ is defined in (37). Using the relation

$$\tanh(x) = 1 + 2 \sum_{k=1}^{\infty} (-1)^k e^{-2kx} \qquad \forall \, x \geqslant 0 \tag{D.3}$$

(imposing $h \geqslant 1$) one has

$$p^{2j} = I_{2j}(0) + 2 \sum_{k=1}^{\infty} (-1)^k e^{-2\beta_s hk} I_{2j}(\beta_s k), \tag{D.4}$$

where $I_{2j}(u)$ are the modified Bessel functions. Then

$$A_n(t) = 2 \sum_{j=1}^{\infty} (-1)^j 2 \sum_{k=1}^{\infty} (-1)^k e^{-2\beta_s hk} I_{2j}(\beta_s k) J_{n-2j}(t). \tag{D.5}$$

The asymptotic behaviour of the Bessel function leads to

$$J_n(t) A_n(t) = \frac{2}{\pi t \sqrt{1 - (n/t)^2}} \sum_{k=1}^{\infty} (-1)^k e^{-2\beta_s hk} 2 \sum_{j=1}^{\infty} (-1)^j I_{2j}(\beta_s k) \cos(2j \arccos(n/t)). \tag{D.6}$$

Since $I_{2j}(u) = (-1)^j J_{2j}(iu)$ one has

$$2 \sum_{j=1}^{\infty} (-1)^j I_{2j}(\beta_s k) \cos(2j \arccos(n/t)) = 2 \sum_{j=1}^{\infty} J_{2j}(i\beta_s k) \cos(2j \arccos(n/t))$$

$$= \cosh(\beta_s k \sqrt{1 - (n/t)^2}) - J_0(i\beta_s k). \tag{D.7}$$

Finally, re-injecting this expression into the previous one, one obtains

$$\Phi'(x, t) = \frac{1}{2\pi t \sqrt{1 - \left(\frac{x}{t}\right)^2}} \left\{ \tanh\left(\frac{\beta_s}{2} \left[h + \sqrt{1 - \left(\frac{x}{t}\right)^2} \right]\right) \right.$$

$$\left. + \tanh\left(\frac{\beta_s}{2} \left[h - \sqrt{1 - \left(\frac{x}{t}\right)^2} \right]\right) \right\}. \tag{D.8}$$

References

[1] Lieb E H, Schultz T D and Mattis D C 1961 *Ann. Phys., NY* **16** 406
[2] Karevski D *Order, Disorder and Criticality* ed Y Holovatch (Singapore: World Scientific)
[3] Brandt U and Jacoby K 1976 *Z. Phys.* B **25** 181
[4] Brandt U and Jacoby K 1977 *Z. Phys.* B **26** 245
[5] Berim G O and Kessel A R 1984 *Teor. Mat. Fiz.* **58** 388
[6] Berim G O 1991 *Sov. J. Low Temp. Phys.* **17** 329
[7] Berim G O 1992 *Sov. Phys.—JETP* **75** 86

 [8] Berim G O and Cabrera G G 1997 *Physica* A **238** 211
 [9] Berim G O, Berim S and Cabrera G G 2002 *Phys. Rev.* B **66** 094401
[10] Schütz G M and Trimper S 1999 *Europhys. Lett.* **47** 164
[11] Iglói F and Rieger H 2000 *Phys. Rev. Lett.* **85** 3233
[12] Karevski D 2002 *Eur. Phys. J.* B **27** 147
[13] Ogata Y 2002 *Phys. Rev.* E **66** 066123
[14] Antal T, Rácz Z and Sasvári L 1997 *Phys. Rev. Lett.* **78** 167
[15] Antal T, Rácz Z, Rákos A and Schütz G M 1998 *Phys. Rev.* E **57** 5184
[16] Antal T, Rácz Z, Rákos A and Schütz G M 1999 *Phys. Rev.* E **59** 4912
[17] Popkov V and Salerno M 2000 *Phys. Rev.* B **62** 352
[18] Eisler V, Rácz Z and van Wijland F 2003 *Phys. Rev.* E **67** 056129-1
[19] Hunyadi V, Rácz Z and Sasvári L 2004 *Phys. Rev.* E **69** 066103
[20] Platini T and Karevski D 2006 *Eur. Phys. J.* B **27** 147
[21] Platini T and Karevski D 2006 *J. Phys. Conf. Ser.* **40** 93
[22] Niemeijer T 1967 *Physica* **36** 377
[23] Tjion J A 1970 *Phys. Rev.* B **2** 2411
[24] Barouch E and Dresden M 1969 *Phys. Rev. Lett.* **23** 114
[25] Barouch E, McCoy B M and Dresden M 1970 *Phys. Rev.* A **1075**
[26] Terwiel R H and Mazur P 1966 *Physica* **32** 1813
[27] Abriet S and Karevski D 2002 *Eur. Phys. J.* B **30** 77
[28] Kac M 1959 *Probability and Related Topics in Physical Sciences* (New York: Interscience)
[29] De Roeck W, Jacobs T, Maes C and Netočný K 2003 *J. Phys. A: Math. Gen.* **36** 11547
[30] Jordan P and Wigner E Z 1928 *Z. Phys.* **47** 631
[31] Gradshteyn I S and Ryszhik I M 1980 *Table of Integrals Series and Products* (London: Academic)